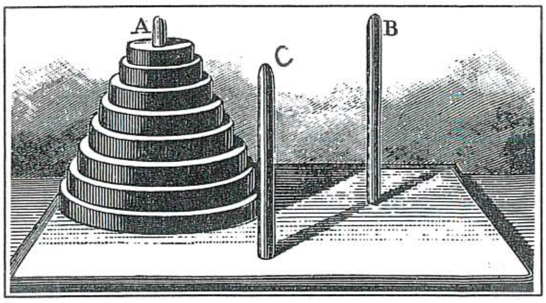
Chap 1. Recurrence

No.
Date: / /

- How to solve a problem
  - (1) Small cases (get a feel for the prob)
  - (2) Find a pattern
  - (3) Guess the solution
  - (4) Prove/derive the solution



• Example 1 (Tower of Hanoi)

n	1	2	3	4	5	6	7
T_n	1	3	7	15	31	63	127

$$\begin{cases} T_1 = 1 \\ T_n = 2T_{n-1}+1 \end{cases} \Rightarrow \text{(甲)}\ T_n = 2^n - 1 \ \text{(by induction)}$$

(乙) $T_n + 1 = 2(T_{n-1}+1) = 2^2(T_{n-2}+1) = \cdots = 2^{n-1}(T_1+1)$
$= 2^n$

(丙) $\dfrac{T_n}{2^n} = \dfrac{T_{n-1}}{2^{n-1}} + \dfrac{1}{2^n}$

$\dfrac{T_{n-1}}{2^{n-1}} = \dfrac{T_{n-2}}{2^{n-2}} + \dfrac{1}{2^{n-1}}$ $\Rightarrow T_n = 2^n\left(\dfrac{1}{2^n} + \dfrac{1}{2^{n-1}} + \cdots + \dfrac{1}{2}\right)$

$\vdots$

$\dfrac{T_2}{2^2} = \dfrac{T_1}{2^1} + \dfrac{1}{2^2}$ $= 1 + 2 + \cdots + 2^{n-1}$
$= 2^n - 1$

• Example 2 (Josephus problem)

(Masada) $n=41$

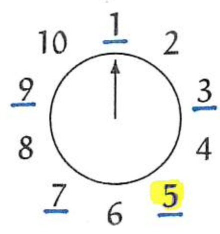
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
J(n)	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

(1) $J(n)$ 奇數

(2) $\begin{cases} J(2^m) = 1 \\ J(2^m - 1) = 2^m - 1, \end{cases}$

(3) $\begin{cases} J(2n) = 2J(n) - 1, & J(10) = 2\cdot \underset{\sim}{3} - 1 \\ J(2n+1) = 2J(n)+1, & J(11) = 2\cdot\underset{\sim}{3}+1 \end{cases}$ ($3 = J(5)$)

$n = 10$

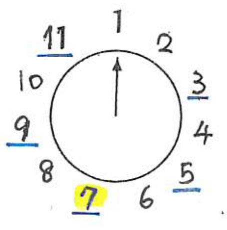


$1 = 2\cdot 1 - 1$
$3 = 2\cdot 2 - 1$
$5 = 2\cdot 3 - 1$
$7 = 2\cdot 4 - 1$
$9 = 2\cdot 5 - 1$

$\Rightarrow J(n) = J(2^m + l) \ (0 \le l < 2^m) = 2l + 1 = 2(n - 2^{\lfloor \lg n \rfloor}) + 1$

$= J(\underset{1}{b_m} b_{m-1} \cdots b_0)_2 = (b_{m-1} \cdots b_0 1)_2$

$n = 11$



$3 = 2\cdot 1 + 1$
$5 = 2\cdot 2 + 1$
$7 = 2\cdot 3 + 1$
$9 = 2\cdot 4 + 1$
$11 = 2\cdot 5 + 1$

$\begin{cases} J(10) = J(1010)_2 = (101)_2 = 5 \\ J(100) = J(1100100)_2 = (1001001)_2 = 73 \end{cases}$

問題 $J(n) = \frac{n}{2}$, $n = ?$
(偶數)

$n = 2 = 10$
$10 = 1010$
$42 = 101010$
$170 = 10101010$

(1) $2l + 1 = \dfrac{2^m + l}{2} \Rightarrow m$ 奇數, (否則 $m = 2k$, $(3+1)^k = 3l + 2$ (*))

(2)
$n = 1\ b_6\ b_5\ b_4\ b_3\ b_2\ b_1\ 0$
$\frac{n}{2} = \quad 1\ b_6\ b_5\ b_4\ b_3\ b_2\ b_1$
$J(n) = b_6\ b_5\ b_4\ b_3\ b_2\ b_1\ 0\ 1$

Generalization

No.
Date: / /

(I) (recurrence, period = 2) (Josephus: $\alpha=1$, $\beta=-1$, $\gamma=1$)

$$f(1) = \alpha$$

$$\begin{cases} f(2n) = 2f(n)+\beta & (n \geq 1) \\ f(2n+1) = 2f(n)+\gamma \end{cases}$$

n	$f(n)$	$f(n)$	n
1	α	α	1
2	$2\alpha+\beta$	$2\alpha+\beta$	10
3	$2\alpha+\gamma$	$2\alpha+\gamma$	11
4	$4\alpha+3\beta$	$4\alpha+2\beta+\beta$	100
5	$4\alpha+2\beta+\gamma$	$4\alpha+2\beta+\gamma$	101
6	$4\alpha+\beta+2\gamma$	$4\alpha+2\gamma+\beta$	110
7	$4\alpha+3\gamma$	$4\alpha+2\gamma+\gamma$	111
8	$8\alpha+7\beta$		
9	$8\alpha+6\beta+\gamma$		
10	$8\alpha+5\beta+2\gamma$	$8\alpha+4\beta+2\gamma+\beta$	1010
11	$8\alpha+4\beta+3\gamma$		

(II) (period = 3)

$$f(1) = 34$$

$$f(2) = 5$$

$$\begin{cases} f(3n) = 10f(n)+76 & (n \geq 1) \\ f(3n+1) = 10f(n)-2 \\ f(3n+2) = 10f(n)+8 \end{cases}$$

$$\begin{aligned} f(46) &= f(1201)_3 \\ &= 10f(120)_3 - 2 \\ &= 10^2 f(12)_3 + 10\cdot 76 - 2 \\ &= 10^3 f(1) + 10^2\cdot 8 + 10\cdot 76 - 2 \\ &= 10^3\cdot 34 + 10^2\cdot 8 + 10\cdot 76 - 2 \\ &= (34, 8, 76, -2)_{10} \end{aligned}$$

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

$$\begin{cases} A(n) = 2^m \\ B(n) = 2^m - 1 - l \\ C(n) = l \end{cases} \xrightarrow[\beta=-1,\ \gamma=1]{\alpha=1} J(n) = 2l+1$$

(Josephus)

(1) Induction: $\quad (1+2+\cdots+n = An^2+Bn+C)$, $n=0,1,2$

(2) Repertoire:

(α,β,γ)	$f(n)$	Eg.
$(1,0,0)$	2^m	$A(n) = 2^m$
$(1,-1,-1)$	1	$A(n)-B(n)-C(n) = 1$
$(1,0,1)$	n	$A(n) + C(n) = n$

- $\begin{cases} f(1) = 1 \\ f(2n) = 2f(n) \\ f(2n+1) = 2f(n) \end{cases}$ $\quad f(n) = 2^m$
- $f(n) = 1$ $\quad \begin{cases} 1 = \alpha \\ 1 = 2+\beta \\ 1 = 2+\gamma \end{cases}$ $\quad (\alpha,\beta,\gamma) = (1,-1,-1)$
- $f(n) = n$ $\quad \begin{cases} 1 = \alpha \\ 2n = 2n+\beta \\ 2n+1 = 2n+\gamma \end{cases}$ $\quad (\alpha,\beta,\gamma) = (1,0,1)$

(3)
$$\begin{aligned} f(1010)_2 &= 2f(101) + \beta \\ &= 2^2 f(10) + 2\gamma + \beta \\ &= 2^3 f(1) + 2^2\beta + 2\gamma + \beta \\ &= 2^3\alpha + 2^2\beta + 2\gamma + \beta \\ &= (\alpha\ \beta\ \gamma\ \beta)_2 \end{aligned}$$